→ Colorings

→ Chromatic Polynomial

→ Deletion-Contraction Formula

→ Stanley Symmetric Chromatic Function

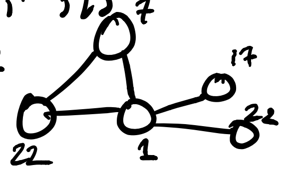
→ Specialization

→ Symmetry

(→ Deletion-Contraction)

Coloring of a graph G : $\chi : V(G) \to \{1, 2, \dots, q\}$

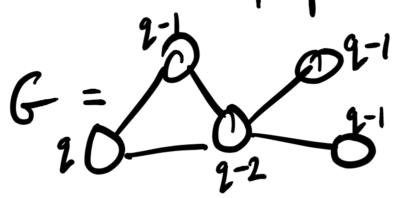
$q \geq 22$



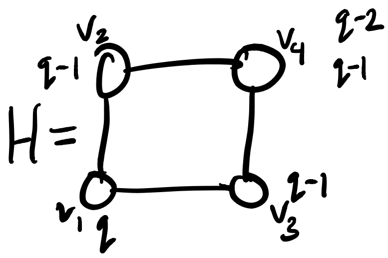
Proper coloring: $\forall$ edges $(v_1, v_2) \in E(G)$

$\chi(v_1) \neq \chi(v_2)$

How to count proper colorings? Chromatic Polynomial



$\chi_G(q) = q(q-1)^3(q-2)$



$\chi_H(q) =$

Case I : $\chi(v_2) = \chi(v_3)$

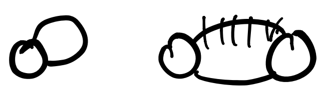
$q(q-1)(q-1)$

Case II : $\chi(v_2) \neq \chi(v_3)$

$q(q-1)(q-2)(q-2)$

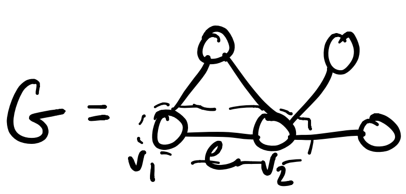
$q(q-1)(q-1) + q(q-1)(q-2)(q-2)$

Special Cases:



Recursive Method: Deletion-Contraction

q^n



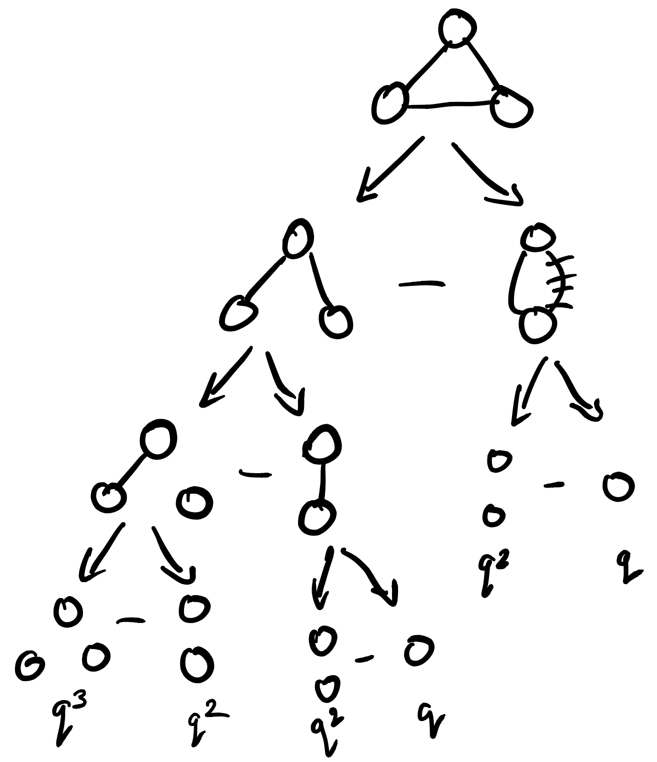
Any proper coloring of G is also a proper coloring $G-e$

However, proper colorings of $G-e$ include cases where $\chi(v_1) = \chi(v_2)$

$\chi(v_1) = \chi(v_2)$

These are colorings when $v_1 = v_2$ (e contracted)

$$\chi_G(q) = \chi_{G-e}(q) - \chi_{G/e}(q)$$



$$\chi_{\triangle}(q) = q^3 - q^2 - (q^2 - q) - (q^2 - q) = q^3 - 3q^2 + 2q$$
$$= q(q-1)(q-2)$$

Stanley Symmetric Chromatic Function

Infinite colors — each color gets 1 variable

$x_1, x_2, \ldots$ etc.

$$X_G(x_1, x_2, \ldots) = \sum_{\chi \text{ proper}} \prod_{v \in V(G)} x_{\chi(v)}$$

$$X_{\triangle}(x_1, x_2, \ldots) = \overset{1}{x_1 x_2 x_3} + \overset{0}{x_1 x_2 x_4} + \overset{0}{x_1 x_2 x_5} + \cdots$$
$$+ \overset{1}{x_1 x_3 x_2} + \overset{0}{x_1 x_3 x_4} + \cdots$$
$$\vdots$$
$$+ \overset{1}{x_2 x_1 x_3} + \overset{1}{x_2 x_3 x_1} + \overset{0}{x_2 x_3 x_4} + \cdots$$
$$\vdots$$
$$= \sum_{\substack{i \neq j \neq k \\ i \neq k}} x_i x_j x_k$$

$X(1^3)$

Symmetric: can switch around variables and nothing changes

$x_{17} \leftrightarrow x_1$

Specialization: For some n, let $\underbrace{x_k = \begin{cases} 1 & k \le n \\ 0 & k > n \end{cases}}_{1^n}$

$$X_G(1^n) = \chi_G(n)$$

$$(\underbrace{1,1,1,1,\dots,1}_{n},0,0,\dots)$$

Symmetric basis: Power function basis p_k

$$p_k = \sum_{i=1}^{\infty} x_i^k \qquad p_1 = x_1 + x_2 + \cdots$$
$$p_5 = x_1^5 + x_2^5 + \cdots$$

$$X_{\triangle}(x_1, x_2, \dots) = \sum_{\substack{i \ne j \ne k \\ i \ne k}} x_i x_j x_k$$

$$= p_1^3 - 3p_1p_2 + 2p_3 \implies n^3 - 3n^2 + 2n$$

$x_i x_j x_k$

(3) $x_i x_j x_j$ (3) $x_i x_j x_j$

(1) $x_i x_i x_i$ (3) $x_1 x_1 x_1$

Note specialization $x_k = 1$ for $k \le n$ is precisely $p_k = n$

$$p_k = x_1^k + x_2^k + x_3^k + \cdots$$
$$= \underbrace{1^k + 1^k + 1^k + \cdots + 1^k}_{n} + 0^k + \cdots$$
$$= n$$